Université de Bourgogne
UFR Sciences et Techniques

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Introduction to TQFT
Exercises

Exercice 1: Let $(A, \mu, 1, \delta, \varepsilon)$ be such that the Frobenius relations, unit and co-unit relations are satisfied. Show that μ is associative and that δ is co-associative.

Exercise 2:

Let A be a finite dimensional $\mathbb{K}$-algebra. We say that a Frobenius form ε is central if it satisfies $\varepsilon(xy) = \varepsilon(yx)$ for all $x, y \in A$. Let ε be a Frobenius form for A.

(1) Show that any Frobenius form on A is of the form $\varepsilon'(x) = \varepsilon(zx)$ where $z \in A$ is invertible.

(2) Show that if ε is central, then any central Frobenius form on A is of the form $\varepsilon'(x) = \varepsilon(zx)$ where $z \in A$ is invertible and central.

Exercise 3:

We want to define a monoidal category $\mathrm{ConnCob}^{2+1}$ such that:

- Objects are compact oriented connected surfaces with one boundary component.
- A morphism $M : S \longrightarrow S'$ is a compact oriented connected manifold M, together with a fixed identification ι of its boundary $\partial M \overset{\iota}{\simeq} \overline{S} \underset{S^1}{\cup} S'$.
- The monoidal product on objects is given by boundary connected sum.

(1) How would you define the composition of morphisms ? the monoidal product on morphisms ? Check that this indeed yields a monoidal category.

(2) What is the monoidal unit ? What are the identity morphisms ? Find a set of objects that monoidally generates (i.e. any object is isomorphic to a product of the generators).

(3) Explain how to associate to any $f \in Diff(S)$ such that $f|_{S^1} = \mathrm{id}_{S^1}$ a *mapping cylinder* morphism $C_f : S \longrightarrow S$.

(4) We define $M_+ : D^2 \longrightarrow \Sigma_{1,1}$ as $D^2 \times S^1$ with a fixed embedding of D^2 in its boundary. Similarly we define a morphism $M_- : \Sigma_{1,1} \longrightarrow D^2$. Find a relation in $\mathrm{ConnCob}^{2+1}$ involving M_+, M_- and a mapping cylinder on $\Sigma_{1,1}$.

(5) Describe as precisely as possible what a monoidal functor $F : \mathrm{ConnCob}^{2+1} \longrightarrow \mathrm{Vect}_{\mathbb{K}}$ should look like.

Exercise 4:

For a link K in S^3, its crossing number $c(K)$ is the minimal number of crossings over all diagrams of K. Its braid index $b(K)$ is the minimal number n such that K can be obtained as the closure of a braid in B_n. We will denote by Tangles the braided monoidal category of framed unoriented tangles.

In this exercise, let F be a braided monoidal functor

$$F : \mathrm{Tangles} \longrightarrow \mathrm{Vect}_{\mathbb{C}}.$$

(1) Explain why any such monoidal functor induces a $\mathbb{C}$-valued invariant of links.

(2) Show that for any knot, one has $b(K) \leq c(K) + 1$.

(3) Show that there exist a constant $A \geq 0$ such that for any knot K, one has $F(K) \leq A^{c(K)}$. (*Hint:* The constant A may be expressed in terms of the coefficient of the image by F of the crossings, the cup and the cap, in a given basis of $V = F(pt)$.)

(4) We now assume that for each n there is an hermitian product $\langle , \rangle$ on $V^{\otimes n}$ such that any element of $F(B_n)$ leaves $\langle , \rangle$ invariant. Show that there is a constant B such that $F(K) \leq B^{b(K)}$, for any knot K.