Université de Bourgogne
UFR Sciences et Techniques

Année 2024-2025

Introduction to TQFT
Midterm exam 13/03/2025

Lecture notes are allowed. All manifolds considered are compact and oriented.

Exercise 1:

Soit $A = \mathbb{C}[x,y]/(x^2,y^2)$ and $\varepsilon : A \longrightarrow \mathbb{C}$ a linear form on A.

(1) Let $a,b,c,d = \varepsilon(1),\varepsilon(x),\varepsilon(y),\varepsilon(xy)$. Express the matrix of the pairing β in terms of a,b,c,d.

(2) Deduce that ε is a Frobenius form if and only if $\varepsilon(xy) \neq 0$.

(3) We now assume that $a,b,c,d = 0,0,0,1$. Compute the co-product on A associated to ε.

Exercise 2:

Let G be a group and let Z be an abelian subgroup of G. We define a category $\mathcal{C}$ by:

- $Obj(\mathcal{C}) = G$.
- For $g,h \in G$, we set $\mathrm{Hom}(g,h) = hZg^{-1}$.
- For $g_1,g_2,g_3 \in G$, $x \in \mathrm{Hom}(g_2,g_3)$ and $y \in \mathrm{Hom}(g_1,g_2)$, their composition is $xy \in \mathrm{Hom}(g_1,g_3)$. (*Here, xy is the product in G*)

(1) Show that the composition is well defined and that this definition indeed gives a category.

(2) Let $\Box$ be defined as follows. For $g,g',h,h' \in \mathrm{Obj}(\mathcal{C})$ we set $g\Box h = gh$ and for $hzg^{-1} \in \mathrm{Hom}(g,h)$ and $h'z'g'^{-1} \in \mathrm{Hom}(g',h')$, we set $(hzg^{-1})\Box(h'z'g'^{-1}) = (hh')zz'(gg')^{-1}$. Show that $(\mathcal{C},\Box)$ is a strict monoidal category, and specify what is the monoidal unit.

(3) For $g,h \in G$ we define $\tau_{g,h} \in \mathrm{Hom}(g\Box h, h\Box g)$ by $\tau_{g,h} = hgh^{-1}g^{-1}$. Show that τ is a symmetric braiding on $(\mathcal{C},\Box)$.

Exercice 3: Let F be a $n+1$-dimensional TQFT over a field $\mathbb{K}$, let Σ,Σ' be closed oriented n-dimensional manifolds, and let $M : \Sigma \longrightarrow \Sigma$ be a cobordism. Let $\beta : \Sigma \coprod \overline{\Sigma} \longrightarrow \emptyset$ be a right U-tube.

Let $\mathcal{B} = (e_1,\ldots,e_n)$ and $\mathcal{B}^* = (e_1^*,\ldots,e_n^*)$ be basis of $F(\Sigma)$ and $F(\overline{\Sigma})$ such that $F(\beta)(e_i,e_j^*) = \delta_{ij}$.

Let $M^* : \overline{\Sigma} \longrightarrow \overline{\Sigma}$ be the cobordism obtained from M by reversing inwards and outwards boundary (while keeping the orientation of M). Show that the matrices of $F(M^*)$ in the basis $\mathcal{B}^*$ is the transpose of the matrix of $F(M)$ in the basis $\mathcal{B}$.

Hint: Write an equivalence of cobordism similar to the snake lemma, but with a cobordism M inserted.

Exercise 4: Let $\mathbb{K}$ be a field. We say that a $\mathbb{K}$-valued invariant of closed oriented n-dimensional manifolds is *multiplicative* if it satisfies $I(M\coprod M') = I(M)I(M')$.

(1) Let F be an $n+1$-TQFT and let I_F be the underlying invariant of closed oriented $n+1$-dimensional manifold. Show that I_F is multiplicative.

From now on, I will denote a multiplicative invariant of closed oriented $n+1$-manifolds.

(2) Let Σ be a closed oriented n-dimensional manifold. Let V_Σ be the $\mathbb{K}$-vector space formally spanned by all cobordisms $M : \emptyset \longrightarrow \Sigma$. We define a bilinear map $\langle , \rangle_\Sigma$ on V_Σ by

$$\langle M, M' \rangle_\Sigma = I(M \underset{\Sigma}{\cup} \overline{M'})$$

when M, M' are cobordisms $\emptyset \longrightarrow \Sigma$ and extend bilinearily. Let $N_\Sigma \subset V_\Sigma$ be the left kernel of $\langle , \rangle_\Sigma$, that is

$$x \in N_\Sigma \Leftrightarrow \forall y \in V_\Sigma, \langle x, y \rangle_\Sigma = 0.$$

For $M : \Sigma \longrightarrow \Sigma'$ a cobordism, we define a linear map $f_M : V_\Sigma \longrightarrow V_{\Sigma'}$ by

$$f_M(M_0) = M \underset{\Sigma}{\cup} M_0$$

when M_0 is a cobordism $\emptyset \longrightarrow \Sigma$. Show that $f_M(N_\Sigma) \subset N_{\Sigma'}$.

(3) For Σ a closed oriented n-dimensional manifold, we set $F_I(\Sigma) = V_\Sigma / N_\Sigma$.

For $M : \Sigma \longrightarrow \Sigma'$ a cobordism, we set $F_I(M) : F_I(\Sigma) \longrightarrow F_I(\Sigma')$ to be the map induced by f_M.

Show that F_I is a functor $\mathrm{Cob}^{n+1} \longrightarrow \mathrm{Vect}_{\mathbb{K}}$.

In the next questions, we will show that F_I is not necessarily a monoidal functor.

(4) Let I be the multiplicative invariant of surfaces such that $I(\Sigma_g) = g$. For $g, b \geq 0$ denote by $\Sigma_{g,b}$ the unique connected cobordism $\emptyset \to \coprod_{1 \leq i \leq b} S^1$ whose underlying surface has genus g and b boundary components. Show that for all $k \geq 2$,

$$\Sigma_{k,1} - k\Sigma_{1,1} + (k-1)\Sigma_{0,1} \in N_{S^1}.$$

(5) Show that $F_I(S^1)$ has dimension 2 and is spanned by $\Sigma_{0,1}$ and $\Sigma_{1,1}$.

(6) Let

$$x = (\Sigma_{1,1} \coprod \Sigma_{0,1}) + (\Sigma_{0,1} \coprod \Sigma_{1,1}) - 2(\Sigma_{0,1} \coprod \Sigma_{0,1}) - \Sigma_{0,2} \in V_{S^1 \coprod S^1}.$$

Show that for any $g_1, g_2 \geq 0$, one has

$$\langle \Sigma_{g_1,1} \coprod \Sigma_{g_2,1}, x \rangle_{S^1 \coprod S^1} = 0.$$

(7) Show that

$$\langle \Sigma_{0,2}, x \rangle \neq 0.$$

Deduce that the $\Sigma_{i,1} \coprod \Sigma_{j,1}$ for $i, j \in \{0, 1\}$ and $\Sigma_{0,2}$ are linearly independent in $F_I(S^1 \coprod S^1)$, and that F_I is not a monoidal functor.