Université de Bourgogne
UFR Sciences et Techniques

Année 2023-2024

Introduction to TQFT
Midterm exam 07/03/2024

All compositions of maps are written using the convention in Kock's book. All manifolds considered are compact and oriented.

Exercice 1:

(1) Let F be a $n+1$-dimensional TQFT over a field $\mathbb{K}$, let M be a n-dimensional manifold, and let $W : M \to M$ be a cobordism. Show that if $F(W)$ is non-invertible, then W is not equivalent to a mapping cylinder.

(2) We now assume that $n=2$. Let Σ_g denote the closed compact oriented surface of genus g. Let S and S' be two surfaces, and let $M : S \to S'$ be a connected $2+1$-cobordism. Show that if $\dim F(\Sigma_g) < rk(F(M))$, then there is no embedding $i : \Sigma_g \longrightarrow M$, such that $M \setminus i(\Sigma_g)$ is disconnected.

Exercise 2:

Let $n \geq 2$ be an integer, let $A = \mathbb{C}[t]/(t^n)$ and let $\varepsilon : A \to \mathbb{C}$ be a linear form on A.

(1) Show that ε is a Frobenius form if and only if $\varepsilon(t^{n-1}) \neq 0$.

(2) For $0 \leq i \leq n-1$, we set $a_i = \varepsilon(t^i)$. Express the matrix of the pairing β in the basis $\{1, t, \ldots, t^{n-1}\}$ in terms of $a_0, \ldots, a_{n-1}$.

(3) Show that the matrix of the co-pairing α is

$$\begin{pmatrix} 0 & \ldots & 0 & b_{n-1} \\ \vdots & \iddots & b_{n-1} & b_{n-2} \\ 0 & \iddots & \iddots & \vdots \\ b_{n-1} & b_{n-2} & \ldots & b_0 \end{pmatrix},$$

where $b_{n-1} = \frac{1}{a_{n-1}}$, and for any $2 \leq i \leq n$,

$$b_{n-i} = -\frac{(a_{n-2}b_{n-i+1} + \ldots + a_{n-i}b_{n-1})}{a_{n-1}}.$$

(4) Let F_A be the $1+1$-TQFT associated to A. Show that the handle element for F_A is

$$w = \frac{nt^{n-1}}{a_{n-1}}.$$

(5) Compute $F_A(\Sigma_g)$ for any connected closed oriented surface of genus $g \geq 0$.

Exercise 3:

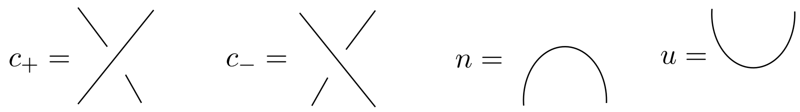
Let $(A, \mu, 1, \Delta, \varepsilon)$ be a Frobenius algebra, and assume that the product is commutative.

(1) Show that $\alpha \circ \tau = \alpha$, where α is the co-pairing and $\tau : A \otimes A \to A \otimes A$ is the twist.

(2) Using the two expressions of Δ in terms of the co-pairing, show that the co-product is cocommutative.

Exercise 4:

In this exercise, we will study the monoidal functors from the category Tangles of tangles in $D^2 \times [0,1]$ to the category $\mathrm{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces. We recall that the category of tangles is generated by the elementary tangles represented in the diagram below:



We also denote as p the object of Tangles consisting of a single point in D^2.

(1) Let F : Tangles $\longrightarrow \mathrm{Vect}_{\mathbb{K}}$ be a monoidal functor. Let $V = F(p), R = F(c_+), R' = F(c_-), \alpha = F(u)$, and $\beta = F(n)$.

Show that the following relations are satisfied:

$$\begin{aligned} RR' = R'R &= \mathrm{id}_{V\otimes V}, & (1)\\ (R \otimes id_V)(id_V \otimes R)(R \otimes id_V) &= (id_V \otimes R)(R \otimes id_V)(id_V \otimes R), & (2)\\ R\beta = \beta, &\qquad \alpha R = \alpha, & (3)\\ (\alpha \otimes id_V)(id_V \otimes \beta) &= id_V = (id_V \otimes \alpha)(\beta \otimes id_V). & (4)\end{aligned}$$

(2) Deduce from Equation (4) above that V is finite dimensional.

(3) We call a link with $n \geq 1$ components an isotopy class of embeddings of $\coprod_{i=1}^{n} S^1$ in the interior of $D^2 \times [0,1]$.

Explain why a monoidal functor F : Tangles $\longrightarrow \mathrm{Vect}_{\mathbb{K}}$ induces a $\mathbb{K}$-valued invariant of links.