Université de Bourgogne
UFR Sciences et Techniques

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Introduction to TQFT
Midterm exam: Solutions

Exercice 1:

We need to show that G is a symmetric monoidal functor, meaning that it respects composition and monoidal product and the symmetric braiding. Let (M, Σ, Σ') and (M', Σ', Σ'') be $n-d+1$-dimension cobordisms. Let us note that the composition of $M \times S$ and $M' \times S$ is the same as $(M \circ M') \times S$. Hence

$$G(M \circ M') = F((M \circ M') \times S) = F((M \times S) \circ (M' \times S)) = F(M \times S) \circ F(M' \times S) = G(M) \circ G(M').$$

Moreover, for Σ an $n-d$-dimensional manifold, we have

$$G(\Sigma \times I) = F((\Sigma \times I) \times S) = F((\Sigma \times S) \times I) = id_{F(\Sigma \times S)} = id_{G(\Sigma)},$$

hence G sends the identity morphism to the identity morphism.

Similarly, for $\emptyset$ the empty $n-d$-manifold, we note that $\emptyset \times S = \emptyset$. Hence $G(\emptyset) = F(\emptyset) = \mathbb{K}$, so G sends the monoidal unit to the monoidal unit. (also, the empty $n-d+1$ cobordism $(\emptyset, \emptyset, \emptyset)$ is sent by G to the identity map $\mathbb{K} \to \mathbb{K}$.)

We also have that for any two $n-d$-manifolds,

$$G(\Sigma \coprod \Sigma') = F((\Sigma \coprod \Sigma') \times S) = F(\Sigma \times S \coprod \Sigma' \times S) = F(\Sigma \times S) \otimes F(\Sigma' \times S) = G(\Sigma) \otimes G(\Sigma'),$$

and similarly for M, M' two $n-d+1$ cobordisms,

$$G(M \coprod M') = F((M \coprod M') \times S) = F(M \times S \coprod M' \times S) = F(M \times S) \otimes F(M' \times S) = G(M) \otimes G(M').$$

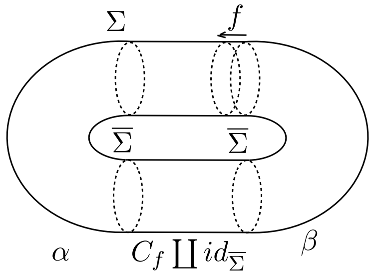
Finally, for Σ, Σ' two $n-d$ dimensional manifold, if $T_{\Sigma,\Sigma'}$ is the *twist* cobordism on Σ, Σ' then $T_{\Sigma,\Sigma'} \times S$ is the twist cobordism on $\Sigma \times S, \Sigma' \times S$. Therefore,

$$G(T_{\Sigma,\Sigma'}) = F(T_{\Sigma \times S, \Sigma' \times S}) = \tau_{F(\Sigma \times S), F(\Sigma' \times S)} = \tau_{G(\Sigma), G(\Sigma')},$$

where for two $\mathbb{K}$-vector spaces V, W the map $\tau_{V,W}$ is the symmetric braiding map: $\tau_{V,W}(v \otimes w) = w \otimes v$.

Exercise 2:

The manifold M_f is diffeomorphic to the following composition of cobordisms:



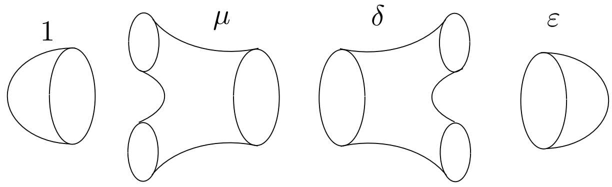
Let $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ be basis of $F(\Sigma)$ and $F(\overline{\Sigma})$. We have seen (as a consequence of the *snake relation*) that $F(\alpha)(1) = \sum_{1 \leq i,j \leq n} a_{i,j} e_i \otimes e'_j$ and that $F(\beta)(e_i, e'_j) = b_{i,j}$, where the matrices A and ${}^t B$ are inverse of each other. Let $M = (m_{ij})$ be the matrix of $\rho(f) = F(C_f)$ in the basis $(e_1, \ldots, e_n)$.

We have that

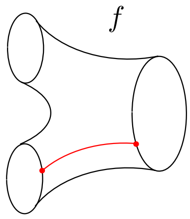
$$F(M_f) = F(\alpha)(F(C_f) \otimes id_{F(\overline{\Sigma})})F(\beta) = \beta\left(\sum_{1 \leq i,j,k \leq n} a_{ij} m_{ki} e_k \otimes e'_j\right)$$
$$= \sum_{1 \leq i,j,k \leq n} a_{ij} m_{ki} b_{kj} = \mathrm{Tr}(A^t BM) = \mathrm{Tr}(M) = \mathrm{Tr}(\rho(f)).$$

Exercise 3:

First, notice that all the usual $1+1$-cobordisms are morphisms in the category $PCob^{1+1}$, as cobordisms containing an empty collection of arcs. Moreover, the relations between those cobordisms that hold in Cob^{1+1} still hold in $PCob^{1+1}$. (one may say that Cob^{1+1} is a *subcategory* of $PCob^{1+1}$). Therefore, one can define a commutative algebra structure on $A = F((S^1, \emptyset))$ using the maps induced by the usual cobordisms:

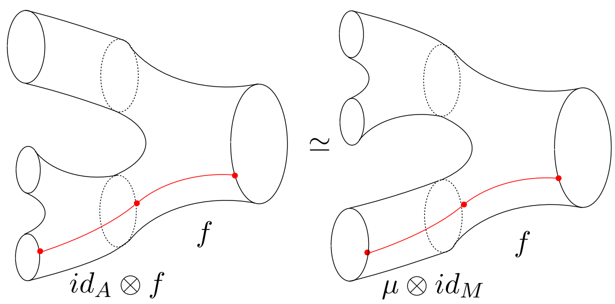


Now, consider the following cobordism in $PCob^{1+1}$ going from $(S^1, \emptyset) \coprod (S^1, \{*\})$ to $(S^1, \{*\})$:



Its image by F is a linear map $f : A \otimes M \to M$. We define the A-module structure on M by $am = f(a \otimes m)$ for $a \in A$ and $m \in M$.

To check that this is indeed a module structure, we need to see if it is compatible with multiplication in A, i.e. $a(bm) = (ab)m$. In terms of the maps f and the multiplication map μ, this is equivalent to the relation $(id_A \otimes f)f = (\mu \otimes id_M)f$. We claim this is the case because we have the following equivalence of cobordisms:



A similar diagram will show that multiplication by the unit induces the identity on M.

Exercise 4:

(1) The closed surface of genus g can be expressed in terms of elementary cobordisms as the composition $1(\Delta\mu)^g\varepsilon$. Since $\Delta\mu$ is the multiplication by w, we get $F_A(\Sigma_g) = \varepsilon(1 \cdot w^g) = \varepsilon(w^g)$.

(2) Since A has finite dimension n, the powers $1, w, \ldots, w^n$ must be linearly dependent. Let d be the least integer such that the powers $1, w, \ldots, w^d$ are linearly dependent, then w^d must be a linear combination of $1, \ldots, w^{d-1}$:

$$w^d = a_0 1 + \ldots + a_{d-1} w^{d-1}.$$

Then, for any $m \geq 0$, we have

$$f(m+d) = \varepsilon(w^{m+d}) = \varepsilon(w^m w^d) = a_0\varepsilon(w^m) + \ldots + a_{d-1}\varepsilon(w^{m+d-1}) = a_0 f(m) + \ldots + a_{d-1} f(m+d-1).$$

(3) ε is a Frobenius form on A if and only if its kernel does not contain any non trivial ideal of A. Since $A = \mathbb{K}$, this happens if and only if $\varepsilon \neq 0$, i.e. $\varepsilon(1) \in \mathbb{K} \setminus \{0\}$, since the kernel of ε is then $\{0\}$.

We have seen that the handle element can be expressed as $\sum_{1 \leq i,j \leq n} a_{i,j} e_i e_j$ if the matrix of the co-pairing is $A = (a_{ij})$ in a basis $\{e_1, \ldots, e_n\}$ of A. Here we choose $1 \in \mathbb{K} = A$ as our basis of A. The matrix of the pairing $\beta(x, y) = \varepsilon(xy)$ is the 1×1 matrix (α), therefore the matrix of the co-pairing is (α^{-1}). Thus $w = \alpha^{-1}$.

Finally, we get

$$f(g) = \varepsilon(w^g) = \varepsilon(\alpha^{-g}) = \alpha^{-g}\varepsilon(1) = \alpha^{1-g}.$$

If $\alpha^n = 1$, then $f(g+n) = f(g)$ and F_A does not distinguish all connected surfaces. If however α is not a root of unity, then $\alpha^{1-g} \neq \alpha^{1-g'}$ for $g \neq g'$ and F_A distinguishes all connected surfaces. However, we have for example

$$F_A(T^2 \coprod T^2) = F_A(T^2)^2 = 1^2 = 1 = F_A(T^2),$$

so F_A does not distinguish the torus from the disjoint union of two tori.

(4) Similarly, let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$, we have $e_i e_j = 0$ if $i \neq j$ and $e_i^2 = e_i$ if $i = j$. The matrix of the pairing has coefficients $\beta_{ij} = \varepsilon(e_i e_j) = 0$ if $i \neq j$, and $\beta_{ii} = \varepsilon(e_i) = \alpha_i$ if $i = j$. This is the diagonal matrix with diagonal $(\alpha_1, \ldots, \alpha_n)$, so the matrix of the co-pairing is the diagonal matrix with diagonal $(\alpha_1^{-1}, \ldots, \alpha_n^{-1})$. Therefore,

$$w = \alpha_1^{-1} e_1 + \ldots + \alpha_n^{-1} e_n = (\alpha_1^{-1}, \ldots, \alpha_n^{-1}).$$

Thus

$$F_A(\Sigma_g) = \varepsilon(w^g) = \varepsilon(\alpha_1^{-g}, \ldots, \alpha_n^{-g}) = \alpha_1^{1-g} + \ldots + \alpha_n^{1-g}.$$

(5) Let Σ and Σ' be two closed orientable surfaces, and let $g_1 \geq g_2 \ldots \geq g_k$ be the genera of the components of Σ, and $g_1' \geq \ldots \geq g_l'$ be the genera of the components of Σ'. The surfaces Σ and Σ' are diffeomorphic if and only if $k = l$ and $g_i = g_i'$ for all i. By monoidality, the invariant of Σ is the product of the invariants of its connected components, and same for Σ'.

Hence if $F_A(\Sigma) = F_A(\Sigma')$ then

$$(\alpha_1^{1-g_1}+\alpha_2^{1-g_1})(\alpha_1^{1-g_2}+\alpha_2^{1-g_2})\dots(\alpha_k^{1-g_k}+\alpha_2^{1-g_k}) = (\alpha_1^{1-g'_1}+\alpha_2^{1-g'_1})(\alpha_1^{1-g'_2}+\alpha_2^{1-g'_2})\dots(\alpha_k^{1-g'_l}+\alpha_2^{1-g'_l}).$$

Since α_1, α_2 are algebraically independent, we have the equality of polynomials in $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$:

$$(X^{1-g_1}+Y^{1-g_1})(X^{1-g_2}+Y^{1-g_2})\dots(X^{1-g_k}+Y^{1-g_k}) = (X^{1-g'_1}+Y^{1-g'_1})(X^{1-g'_2}+Y^{1-g'_2})\dots(X^{1-g'_l}+Y^{1-g'_l}).$$

If for instance $g_1 > g'_1$ and $g_1 \geq 3$ then the left hand side is zero when evaluated at $(1, \zeta)$ where $\zeta = e^{\frac{i\pi}{g_1-1}}$, while the right hand side is non zero. If one of the two surface contains a connected component of genus at least 3, the other has the same maximal genus among its components; then we can erase those components from Σ and Σ' while keeping $F_A(\Sigma) = F_A(\Sigma')$. Therefore we assume that both contain only components of genus $0, 1$ or 2. Let n_0, n_1, n_2 be the number of components of genus $0, 1, 2$ for Σ, and similarly n'_0, n'_1, n'_2 for Σ'. The equality $F_A(\Sigma) = F_A(\Sigma')$ can now be rewritten

$$2^{n_1}(X+Y)^{n_0}(X^{-1}+Y^{-1})^{n_2} = 2^{n'_1}(X+Y)^{n'_0}(X^{-1}+Y^{-1})^{n'_2}.$$

Evaluating at $X = Y = 1$ we get $n_0 + n_1 + n_2 = n'_0 + n'_1 + n'_2$, looking at the maximal degree in X we get $n_0 = n'_0$ and looking at the total degree we get $n_0 - n_2 = n'_0 - n'_2$. Therefore $n_i = n'_i$ for $i = 0, 1, 2$ and $\Sigma \simeq \Sigma'$.