Université de Bourgogne
UFR Sciences et Techniques

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Introduction to TQFT
Midterm exam 13/03/2025

Lecture notes are allowed. All manifolds considered are compact and oriented.

Exercise 1:

Soit $A = \mathbb{C}[x,y]/(x^2,y^2)$ and $\varepsilon : A \longrightarrow \mathbb{C}$ a linear form on A.

(1) Let $a,b,c,d = \varepsilon(1), \varepsilon(x), \varepsilon(y), \varepsilon(xy)$. Express the matrix of the pairing β in terms of a,b,c,d.

Solution: The pairing is the map such that for $z,t \in A$, $\beta(z,t) = \varepsilon(zt)$. Therefore its matrix in the basis $\{1,x,y,xy\}$ is

$$B = \mathrm{Mat}_{\{1,x,y,xy\}}(\beta) = \begin{pmatrix} a & b & c & d \\ b & 0 & d & 0 \\ c & d & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}.$$

(2) Deduce that ε is a Frobenius form if and only if $\varepsilon(xy) \neq 0$.

Solution: ε is a Frobenius form if and only if the associated pairing is non-degenerate, that is, if and only if $B = \mathrm{Mat}(\beta)$ is invertible. Since $\det B = -d^4$, this is the case iff $\varepsilon(xy) \neq 0$.

(3) We now assume that $a,b,c,d = 0,0,0,1$. Compute the co-product on A associated to ε.

Solution: We first compute the matrix A of the co-pairing α :

$$A = {}^tB^{-1} = {}^t\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

hence $\alpha(1) = 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1$. Now, we apply the formula $\Delta = (\mu \otimes id_A)(id_A \otimes \alpha)$. For $t \in A$, we get $\Delta(t) = (\mu \otimes id_A)(t \otimes (1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1)) = t \otimes xy + tx \otimes y + ty \otimes x + txy \otimes 1$. In particular,

$$\begin{aligned} \Delta(1) &= 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1 \\ \Delta(x) &= x \otimes xy + xy \otimes x \\ \Delta(y) &= y \otimes xy + xy \otimes y \\ \Delta(xy) &= xy \otimes xy. \end{aligned}$$

Exercise 2:

Let G be a group and let Z be an abelian subgroup of G. We define a category $\mathcal{C}$ by:

- $Obj(\mathcal{C}) = G$.
- For $g,h \in G$, we set $\mathrm{Hom}(g,h) = hZg^{-1}$.

- For $g_1, g_2, g_3 \in G$, $x \in \mathrm{Hom}(g_2, g_3)$ and $y \in \mathrm{Hom}(g_1, g_2)$, their composition is $xy \in \mathrm{Hom}(g_1, g_3)$. (*Here, xy is the product in G*)

(1) Show that the composition is well defined and that this definition indeed gives a category.

Solution: First, we note that the composition of $x \in \mathrm{Hom}(g_2, g_3)$ and $y \in \mathrm{Hom}(g_1, g_2)$ is indeed in $\mathrm{Hom}(g_1, g_3)$. Writing $x = g_3 z g_2^{-1}$ and $y = g_2 z' g_1^{-1}$ where $z, z' \in Z$, we have

$$xy = g_3 z z' g_1^{-1} \in \mathrm{Hom}(g_1, g_3),$$

since Z is a subgroup.

The associativity of composition follows directly from the associativity of the group law.

Let e be the identity element of G. For any $g \in \mathrm{Obj}(\mathcal{C})$, one has $e = geg^{-1} \in \mathrm{Hom}(g, g)$. Furthermore, since e is an identity element, for h another object of $\mathcal{C}$ one has $ex = x$ for all $x \in \mathrm{Hom}(g, h)$, and $xe = x$ for all $x \in \mathrm{Hom}(h, g)$. Therefore e plays the role of an identity morphism, and $\mathcal{C}$ is a category.

(2) Let $\Box$ be defined as follows. For $g, g', h, h' \in \mathrm{Obj}(\mathcal{C})$ we set $g \Box h = gh$ and for $hzg^{-1} \in \mathrm{Hom}(g, h)$ and $h'z'g'^{-1} \in \mathrm{Hom}(g', h')$, we set $(hzg^{-1}) \Box (h'z'g'^{-1}) = (hh')zz'(gg')^{-1}$. Show that $(\mathcal{C}, \Box)$ is a strict monoidal category, and specify what is the monoidal unit.

Solution: We first check that $\Box$ is a functor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, that is, that $\Box$ is compatible with composition. For $g, g', h, h', t, t' \in \mathrm{Obj}(\mathcal{C})$ and $f_1 = hz_1g^{-1}, f_2 = h'z_2g'^{-1}, f_3 = tz_3h^{-1}, f_4 = t'z_4h'^{-1}$ in $\mathrm{Hom}(g, h), \mathrm{Hom}(g', h'), \mathrm{Hom}(h, t), \mathrm{Hom}(h', t')$, one has

$$\begin{aligned}(f_3 \Box f_4)(f_1 \Box f_2) &= (tz_3h^{-1} \Box t'z_4h'^{-1})(hz_1g^{-1} \Box h'z_2g'^{-1}) = (tt'z_3z_4(hh')^{-1})(hh'z_1z_2(gg')^{-1}) \\ &= tt'z_3z_4z_1z_2(gg')^{-1} = tt'z_3z_1z_4z_2(gg')^{-1} = tz_3z_1g^{-1} \Box t'z_4z_2g'^{-1} = f_3f_1 \Box f_4f_2\end{aligned}$$

where the fourth equality uses that Z is abelian.

Moreover, we have for $g, h \in G$, $id_g \Box id_h = geg^{-1} \Box heh^{-1} = (gh)e(gh)^{-1} = id_{gh}$.

The associativity of $\Box$ on $\mathrm{Obj}(\mathcal{C})$ and on morphisms is clear and follows from the associativity of the product in G. Moreover, $e \Box g = g \Box e = g$ for any $g \in \mathrm{Obj}(\mathcal{C})$, since e is the identity element in $\mathcal{C}$. Furthermore, for $g, h \in \mathrm{Obj}(\mathcal{C})$ and $x = hzg^{-1} \in \mathrm{Hom}(g, h)$, one has $id_e \Box x = (eh)ez(eg)^{-1} = hzg^{-1} = x$. Similarly, $x \Box id_e = x$. So e is a monoidal unit.

(3) For $g, h \in G$ we define $\tau_{g,h} \in \mathrm{Hom}(g \Box h, h \Box g)$ by $\tau_{g,h} = hgh^{-1}g^{-1}$. Show that τ is a symmetric braiding on $(\mathcal{C}, \Box)$.

Solution: First, we check that τ is natural. For $g, g', h, h' \in \mathrm{Obj}(\mathcal{C})$, and $f_1 = g'z_1g^{-1}, f_2 = h'z_2h^{-1}$ in $\mathrm{Hom}(g, g')$ and $\mathrm{Hom}(h, h')$ respectively, we have

$$\begin{aligned}(f_2 \Box f_1)\tau_{g,h} &= (h'z_2h^{-1} \Box g'z_1g^{-1})hgh^{-1}g^{-1} = (h'g'z_2z_1(hg)^{-1})hgh^{-1}g^{-1} \\ &= (h'g'h'^{-1}g'^{-1})(g'h'z_1z_2(gh)^{-1}) = \tau_{g',h'}(f_1 \Box f_2).\end{aligned}$$

Next, we check that τ satisfies the braiding relations. For $g, h, t \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned}(id_h \Box \tau_{g,t})(\tau_{g,h} \Box id_t) &= (heh^{-1} \Box tget^{-1}g^{-1})(hgeh^{-1}g^{-1} \Box tet^{-1}) \\ &= (htgt^{-1}g^{-1}h^{-1})(hgh^{-1}g^{-1}) = (ht)g(ht)^{-1}g^{-1} = \tau_{g,ht} = \tau_{g,h \Box t}\end{aligned}$$

Similarly, one has $\tau_{g\Box h,t} = (\tau_{g,t}\Box id_h)(id_g\Box\tau_{h,t})$.

Finally, we show that τ is symmetric. For $g, h \in \mathrm{Obj}(\mathcal{C})$, one has

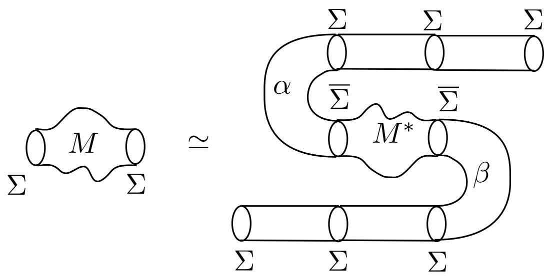
$$\tau_{g,h}\tau_{h,g} = (hgh^{-1}g^{-1})(ghg^{-1}h^{-1}) = e = id_{h\Box g}.$$

Exercice 3: Let F be a $n+1$-dimensional TQFT over a field $\mathbb{K}$, let Σ, Σ' be closed oriented n-dimensional manifolds, and let $M : \Sigma \longrightarrow \Sigma$ be a cobordism. Let $\beta : \Sigma \coprod \overline{\Sigma} \longrightarrow \emptyset$ be a right U-tube.

Let $\mathcal{B} = (e_1, \ldots, e_n)$ and $\mathcal{B}^* = (e_1^*, \ldots, e_n^*)$ be basis of $F(\Sigma)$ and $F(\overline{\Sigma})$ such that $F(\beta)(e_i, e_j^*) = \delta_{ij}$.

Let $M^* : \overline{\Sigma} \longrightarrow \overline{\Sigma}$ be the cobordism obtained from M by reversing inwards and outwards boundary (while keeping the orientation of M). Show that the matrices of $F(M^*)$ in the basis $\mathcal{B}^*$ is the transpose of the matrix of $F(M)$ in the basis $\mathcal{B}$.

Solution: Considering a collar of the boundary of M, one decompose it in several cylinders and U-tubes and get the following equivalence of cobordisms:



Hence, applying F we have

$$F(M) = (F(\beta) \otimes id_{F(\Sigma)})(id_{F(\Sigma)} \otimes F(M^*) \otimes id_{F(\Sigma)})(id_{F(\Sigma)} \otimes F(\alpha)),$$

where $\alpha : \emptyset \longrightarrow \overline{\Sigma} \coprod \Sigma$ is a left U-tube. Now the snake lemma gives

$$F(\alpha)(1) = \sum_{1\leq i\leq n} e_i^* \otimes e_i.$$

Let A be the matrix of $F(M^*)$ in the basis $\mathcal{B}^*$. We have

$$\begin{aligned} F(M)e_i &= (F(\beta) \otimes id_{F(\Sigma)})(id_{F(\Sigma)} \otimes F(M^*) \otimes id_{F(\Sigma)})(e_i \otimes \sum_{1\leq j\leq n} e_j^* \otimes e_j) \\ &= (F(\beta) \otimes id_{F(\Sigma)})(e_i \otimes \sum_{1\leq j,k\leq n} A_{kj} e_k^* \otimes e_j) = \sum_{1\leq j\leq n} A_{ij} e_j \end{aligned}$$

which means that the matrix of $F(M)$ in the basis $\mathcal{B}$ is ${}^tA$.

Exercise 4: Let $\mathbb{K}$ be a field. We say that a $\mathbb{K}$-valued invariant of closed oriented n-dimensional manifolds is *multiplicative* if it satisfies $I(M \coprod M') = I(M)I(M')$.

(1) Let F be an $n+1$-TQFT and let I_F be the underlying invariant of closed oriented $n+1$-dimensional manifold. Show that I_F is multiplicative.

Solution: A closed $n+1$-manifold M is a cobordism $\emptyset \longrightarrow \emptyset$, and one has $F(\emptyset) \simeq \mathbb{K}$ and $F(M) \in \mathrm{End}(\mathbb{K})$ is by the definition the scalar multiplication by $I_F(M)$. By monoidality, for M, M' closed $n+1$-manifolds, $F(M \coprod M') \simeq F(M) \otimes F(M')$ is the multiplication by $I_F(M)I_F(M')$, i.e. $I_F(M \coprod M') = I_F(M)I_F(M')$ and I_F is multiplicative.

From now on, I will denote a multiplicative invariant of closed oriented $n+1$-manifolds.

(2) Let Σ be a closed oriented n-dimensional manifold. Let V_Σ be the $\mathbb{K}$-vector space formally spanned by all cobordisms $M : \emptyset \longrightarrow \Sigma$. We define a bilinear map $\langle , \rangle_\Sigma$ on V_Σ by

$$\langle M, M' \rangle_\Sigma = I(M \underset{\Sigma}{\cup} \overline{M'})$$

when M, M' are cobordisms $\emptyset \longrightarrow \Sigma$ and extend bilinearily. Let $N_\Sigma \subset V_\Sigma$ be the left kernel of $\langle , \rangle_\Sigma$, that is

$$x \in N_\Sigma \Leftrightarrow \forall y \in V_\Sigma, \langle x, y \rangle_\Sigma = 0.$$

For $M : \Sigma \longrightarrow \Sigma'$ a cobordism, we define a linear map $f_M : V_\Sigma \longrightarrow V_{\Sigma'}$ by

$$f_M(M_0) = M \underset{\Sigma}{\cup} M_0$$

when M_0 is a cobordism $\emptyset \longrightarrow \Sigma$. Show that $f_M(N_\Sigma) \subset N_{\Sigma'}$.

Solution: Let $x = \sum_i \lambda_i M_i \in N_\Sigma$. Then for any cobordism $M' : \emptyset \longrightarrow \Sigma'$, one has

$$\begin{aligned}\langle f_M(x), M' \rangle_{\Sigma'} = \sum_i \lambda_i \langle M_i \underset{\Sigma}{\cup} M, M' \rangle_{\Sigma'} &= \sum_i \lambda_i I(M_i \underset{\Sigma}{\cup} M \underset{\Sigma'}{\cup} \overline{M'}) \\ &= \sum_i \lambda_i \langle M_i, \overline{M} \underset{\Sigma'}{\cup} M' \rangle_\Sigma = \langle x, \overline{M} \underset{\Sigma'}{\cup} M' \rangle_\Sigma = 0\end{aligned}$$

since $x \in N_\Sigma$. By bilinearity of $\langle , \rangle_{\Sigma'}$, we deduce that $f_M(x) \in N_{\Sigma'}$.

(3) For Σ a closed oriented n-dimensional manifold, we set $F_I(\Sigma) = V_\Sigma / N_\Sigma$.

For $M : \Sigma \longrightarrow \Sigma'$ a cobordism, we set $F_I(M) : F_I(\Sigma) \longrightarrow F_I(\Sigma')$ to be the map induced by f_M.

Show that F_I is a functor $\mathrm{Cob}^{n+1} \longrightarrow \mathrm{Vect}_\mathbb{K}$.

Solution: It is clear that $F_I(\Sigma)$ is a $\mathbb{K}$-vector space and by (2), the map f_M induces a linear map $F_I(\Sigma) \longrightarrow F_I(\Sigma')$. It remains to check that F_I sends compositions to compositions. However, for $M : \Sigma \longrightarrow \Sigma'$ and $M' : \Sigma' \longrightarrow \Sigma''$, we have $f_{M'} \circ f_M = f_{M' \underset{\Sigma'}{\cup} M}$. Indeed, for any $M_0 : \emptyset \longrightarrow \Sigma$, we have

$$f_{M'}(f_M(M_0)) = f_{M'}(M \underset{\Sigma}{\cup} M_0) = (M' \underset{\Sigma'}{\cup} M) \underset{\Sigma}{\cup} M_0.$$

The same is then true for the maps $F_I(M)$.

Finally, we check that $F_I(\Sigma \times [0,1])(M_0) = M_0$ for all cobordism $M_0 : \emptyset \longrightarrow \Sigma$, since $\Sigma \times [0,1] \underset{\Sigma}{\cup} M \simeq M$ and I is a diffeomorphism invariant. So F_I sends identity morphisms to identity morphisms.

In the next questions, we will show that F_I is not necessarily a monoidal functor.

(4) Let I be the multiplicative invariant of surfaces such that $I(\Sigma_g) = g$. For $g, b \geq 0$ denote by $\Sigma_{g,b}$ the unique connected cobordism $\emptyset \to \coprod_{1\leq i\leq b} S^1$ whose underlying surface has genus g and b boundary components. Show that for all $k \geq 2$,

$$\Sigma_{k,1} - k\Sigma_{1,1} + (k-1)\Sigma_{0,1} \in N_{S^1}.$$

Solution: We compute for each $g \geq 1$,

$$\begin{aligned}\langle \Sigma_{k,1} - k\Sigma_{1,1} + (k-1)\Sigma_{0,1}, \Sigma_{g,1}\rangle_{S^1} &= I(\Sigma_{g+k}) - kI(\Sigma_{g+1}) + (k-1)I(\Sigma_g) \\ &= (g+k) - k(g+1) + (k-1)g = 0\end{aligned}$$

Hence, by bilinearity of $\langle , \rangle_{S^1}$ and since V_{S^1} is spanned by the $\Sigma_{g,1}$, we have that $\Sigma_{k,1} - k\Sigma_{1,1} + (k-1)\Sigma_{0,1} \in N_{S^1}$.

(5) Show that $F_I(S^1)$ has dimension 2 and is spanned by $\Sigma_{0,1}$ and $\Sigma_{1,1}$.

Solution: The identity obtained in (4) implies that $F_I(S^1)$ is spanned by $\Sigma_{0,1}$ and $\Sigma_{1,1}$. It remains to show that $\Sigma_{0,1}$ and $\Sigma_{1,1}$ are linearly independent in $F_I(S^1)$.

Assume that $a\Sigma_{0,1} + b\Sigma_{1,1} \in N_{S^1}$ for some constant $a, b \in \mathbb{K}$. Then $0 = \langle a\Sigma_{0,1} + b\Sigma_{1,1}, \Sigma_{0,1}\rangle = aI(\Sigma_0) + bI(\Sigma_1) = b$, and $0 = \langle a\Sigma_{0,1} + b\Sigma_{1,1}, \Sigma_{1,1}\rangle = aI(\Sigma_1) + bI(\Sigma_2) = a + 2b$. Hence $a = b = 0$. So $\Sigma_{0,1}$ and $\Sigma_{1,1}$ are linearly independent.

Another way to show this is to remark that the matrix $A = (\langle \Sigma_{i,1}, \Sigma_{j,1}\rangle)_{0\leq i,j\leq 1} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ is invertible, which implies that $\langle , \rangle_{S^1}$ is non-degenerate in restriction to $\mathrm{Span}(\Sigma_{0,1}, \Sigma_{1,1})$.

(6) Let

$$x = (\Sigma_{1,1} \coprod \Sigma_{0,1}) + (\Sigma_{0,1} \coprod \Sigma_{1,1}) - 2(\Sigma_{0,1} \coprod \Sigma_{0,1}) - \Sigma_{0,2} \in V_{S^1 \coprod S^1}.$$

Show that for any $g_1, g_2 \geq 0$, one has

$$\langle \Sigma_{g_1,1} \coprod \Sigma_{g_2,1}, x\rangle_{S^1 \coprod S^1} = 0.$$

Solution: We compute

$$\begin{aligned}\langle \Sigma_{g_1,1} \coprod \Sigma_{g_2,1}, x\rangle_{S^1 \coprod S^1} &= I(\Sigma_{g_1+1})I(\Sigma_{g_2}) + I(\Sigma_{g_1})I(\Sigma_{g_2+1}) - 2I(\Sigma_{g_1})I(\Sigma_{g_2}) - I(\Sigma_{g_1+g_2}) \\ &= (g_1+1)g_2 + g_1(g_2+1) - 2g_1g_2 - (g_1+g_2) = 0.\end{aligned}$$

(7) Show that

$$\langle \Sigma_{0,2}, x\rangle \neq 0.$$

Deduce that the $\Sigma_{i,1} \coprod \Sigma_{j,1}$ for $i, j \in \{0, 1\}$ and $\Sigma_{0,2}$ are linearly independent in $F_I(S^1 \coprod S^1)$, and that F_I is not a monoidal functor.

Solution: Let B be the matrix whose entries are parametrized by $0 \leq i, j, k, l \leq 1$ and whose coefficients in line (i,j) column (k,l) is $\langle \Sigma_{i,1} \coprod \Sigma_{j,1}, \Sigma_{k,1} \coprod \Sigma_{l,1}\rangle_{S^1 \coprod S^1}$. Then $B = A \otimes A$, where A is the matrix introduced in (5), and therefore B is invertible. It follows that the $\Sigma_{i,1} \coprod \Sigma_{j,1}$ for $0 \leq i, j \leq 1$ are linearly independent in $F_I(S^1 \coprod S^1)$. Furthermore,

$$\langle \Sigma_{0,2}, x \rangle = I(\Sigma_1) + I(\Sigma_1) - 2I(\Sigma_0) - I(\Sigma_1) = 1 + 1 - 2 \cdot 0 - 1 = 1 \neq 0.$$

Therefore, comparing with (6), we get that as elements of $F_I(S^1 \coprod S^1)$, one has

$$\Sigma_{0,2} \notin \mathrm{Span}\{\Sigma_{i,1} \coprod \Sigma_{j,1} | i, j \geq 0\}.$$

Thus we have found 5 linearly independent element in $F_I(S^1 \coprod S^1)$ and thus

$$\dim F_I(S^1 \coprod S^1) \leq 5 > 4 = \dim F_I(S^1) \otimes F_I(S^1).$$

Hence F_I is not a monoidal functor.

Remark: One may however show that $F_I(\Gamma)$ is finite dimensional, for any closed 1-dimensional manifold Γ.