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UFR Sciences et Techniques

Introduction to TQFT

Exercice 1:

(1) Let F be a $n+1$-dimensional TQFT over a field $\mathbb{K}$, let M be a n-dimensional manifold, and let $W : M \to M$ be a cobordism. Show that if $F(W)$ is non-invertible, then W is not equivalent to a mapping cylinder.

Solution: Recall that for $f, g \in \mathrm{Diff}(M)$, we have $C_f \circ C_g$ is equivalent to C_{fg}, where C_f is the mapping cylinder of f. Hence

$$F(C_f) \circ F(C_{f^{-1}}) = F(C_{id_M}) = id_{F(M)},$$

hence $F(C_f)$ is invertible. Therefore if $F(W)$ is not invertible, W is not equivalent to a mapping cylinder.

(2) We now assume that $n = 2$. Let Σ_g denote the closed compact oriented surface of genus g. Let S and S' be two surfaces, and let $M : S \to S'$ be a connected $2+1$-cobordism. Show that if $\dim F(\Sigma_g) < rk(F(M))$, then there is no embedding $i : \Sigma_g \longrightarrow M$, such that $M \setminus i(\Sigma_g)$ is disconnected.

Solution: We will prove it by contraposition. Consider M_1 (resp. M_2) the connected component of $M \setminus i(\Sigma_g)$ containing S, S'. Make M_1, M_2 into cobordisms $S \longrightarrow \Sigma_g, \Sigma_g \longrightarrow S'$ using i as identification map. Then $M = M_1 \circ M_2$, and $F(M) = F(M_1) \circ F(M_2)$, which implies that $rk(F(M)) \leq rk(F(M_1)) \leq \dim F(\Sigma_g)$.

Exercise 2:

Let $n \geq 2$ be an integer, let $A = \mathbb{C}[t]/(t^n)$ and let $\varepsilon : A \to \mathbb{C}$ be a linear form on A.

(1) Show that ε is a Frobenius form if and only if $\varepsilon(t^{n-1}) \neq 0$.

(2) For $0 \leq i \leq n-1$, we set $a_i = \varepsilon(t^i)$. Express the matrix of the pairing β in the basis $\{1, t, \ldots, t^{n-1}\}$ in terms of $a_0, \ldots, a_{n-1}$.

Solution to (1) and (2): Note that $1, t, \ldots, t^{n-1}$ is a basis of A. For $0 \leq i, j \neq n-1$, we have

$$\beta(t^i, t^j) = \varepsilon(t^{i+j}) = \begin{cases} 0 \text{ if } i+j \geq n \\ a_{i+j} \text{ else} \end{cases}.$$

Therefore, the matrix of β is

$$\begin{pmatrix} a_0 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \iddots & \iddots & 0 \\ a_{n-2} & a_{n-1} & \iddots & \vdots \\ a_{n-1} & 0 & \cdots & 0 \end{pmatrix}$$

Its determinant is $(-1)^n a_{n-1}^n$, hence it is invertible if and only if $a_{n-1} \neq 0$, so ε is a Frobenius form iff $a_{n-1} \neq 0$.

Alternatively, for (1), one could show that the only ideals in A are $A, tA, \ldots t^{n-1}A$.

(3) Show that the matrix of the co-pairing α is

$$\begin{pmatrix} 0 & \dots & 0 & b_{n-1} \\ \vdots & \iddots & b_{n-1} & b_{n-2} \\ 0 & \iddots & \iddots & \vdots \\ b_{n-1} & b_{n-2} & \dots & b_0 \end{pmatrix},$$

where $b_{n-1} = \frac{1}{a_{n-1}}$, and for any $2 \le i \le n$,

$$b_{n-i} = -\frac{(a_{n-2}b_{n-i+1} + \ldots + a_{n-i}b_{n-1})}{a_{n-1}}.$$

Solution: For $b_0, \ldots, b_{n-1} \in \mathbb{C}$, we have that:

$$\begin{pmatrix} a_0 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \iddots & \iddots & 0 \\ a_{n-2} & a_{n-1} & \iddots & \vdots \\ a_{n-1} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 & b_{n-1} \\ \vdots & \iddots & b_{n-1} & b_{n-2} \\ 0 & \iddots & \iddots & \vdots \\ b_{n-1} & b_{n-2} & \dots & b_0 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & \dots & c_{n-1} \\ 0 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_1 \\ 0 & \dots & 0 & c_0 \end{pmatrix},$$

where $c_0 = a_{n-1}b_{n-1}$ and for $i \ge 2$, $c_{i-1} = a_{n-1}b_{n-i} + a_{n-2}b_{n-i+1} \ldots + a_{n-i}b_{n-1}$.

So the matrices are inverse of each other if and only if $a_{n-1}b_{n-1} = 1$ and for $2 \le i \le n$, we have

$$a_{n-1}b_{n-i} + a_{n-2}b_{n-i+1} = 0,$$

which is equivalent to the identities asked.

(4) Let F_A be the $1+1$-TQFT associated to A. Show that the handle element for F_A is

$$w = \frac{nt^{n-1}}{a_{n-1}}.$$

Solution: We use the expression of the handle element in terms of the matrix $(\alpha_{i,j})_{0 \le i,j \le n-1}$ of the co-pairing α in the basis $1, t, \ldots, t^{n-1}$. The handle element is

$$w = \sum_{0 \le i,j \le n-1} \alpha_{i,j} t^i t^j = nb_{n-1}t^{n-1} = \frac{nt^{n-1}}{a_{n-1}},$$

since $t^{i+j} = 0$ if $i + j > n - 1$, and $\alpha_{i,j} = 0$ if $i + j < n - 1$, and $\alpha_{i,j} = b_{n-1}$ if $i + j = n - 1$.

(5) Compute $F_A(\Sigma_g)$ for any connected closed oriented surface of genus $g \ge 0$.

Solution: Recall that a surface of genus g is the cobordism $1 \circ (\Delta \circ \mu)^g \varepsilon$, and $\Delta \circ \mu$ is the multiplication by the handle element w. Therefore, $F_A(\Sigma_g) = \varepsilon(w^g)$.

By question (4):

$$F_A(\Sigma_g) = \begin{cases} \varepsilon(1) = a_0 \text{ if } g = 0 \\ \varepsilon(w) = \varepsilon(\frac{nt^{n-1}}{a_{n-1}}) = n \text{ if } g = 1 \\ \varepsilon(w^g) = \varepsilon(0) = 0 \text{if } g \ge 2. \end{cases}$$

Exercise 3:

Let $(A, \mu, 1, \Delta, \varepsilon)$ be a Frobenius algebra, and assume that the product is commutative.

(1) Show that $\alpha \circ \tau = \alpha$, where α is the co-pairing and $\tau : A \otimes A \to A \otimes A$ is the twist.

Solution: Let $e_1, \ldots, e_n$ be a basis of A. Since the product is commutative, then $\beta(e_i, e_j) = \varepsilon(e_i e_j) = \varepsilon(e_j e_i) = \beta(e_j, e_i)$. Therefore the matrix of the pairing is symmetric. Hence, the matrix of the co-pairing α is symmetric, since it is the transpose of the inverse of the matrix of β.

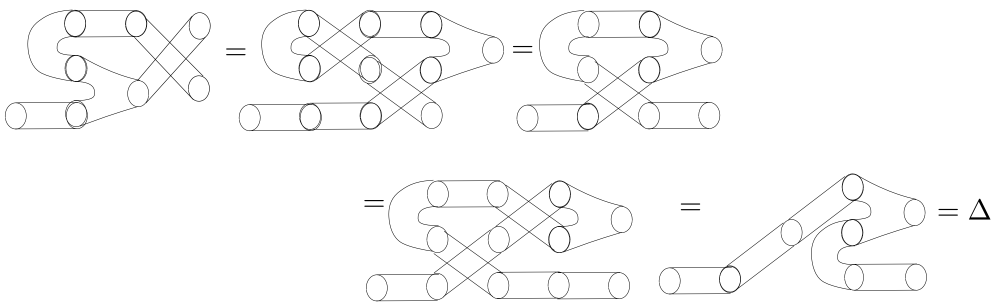
Then

$$(1)\alpha \circ \tau = \left(\sum_{i,j} \alpha_{i,j} e_i \otimes e_j \right) \tau = \sum_{i,j} \alpha_{i,j} e_j \otimes e_i = \sum_{i,j} \alpha_{i,j} e_i \otimes e_j = (1)\alpha.$$

(2) Using the two expressions of Δ in terms of the co-pairing, show that the co-product is cocommutative.

Solution:

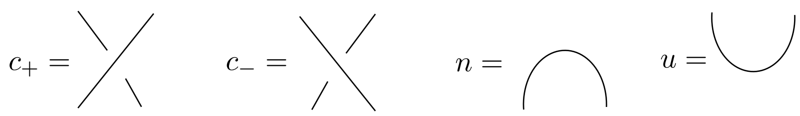
We need to show that $\Delta \circ \tau = \Delta$, which we do by graphical calculus as follows:



Note that in the first and fourth equality, we used the naturality of the twist, in the second the relation $\alpha \circ \tau = \alpha$, and in the third, the relation $\tau \circ \mu = \mu$, as the product is commutative.

Exercise 4:

In this exercise, we will study the monoidal functors from the category Tangles of tangles in $D^2 \times [0,1]$ to the category $\mathrm{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces. We recall that the category of tangles is generated by the elementary tangles represented in the diagram below:



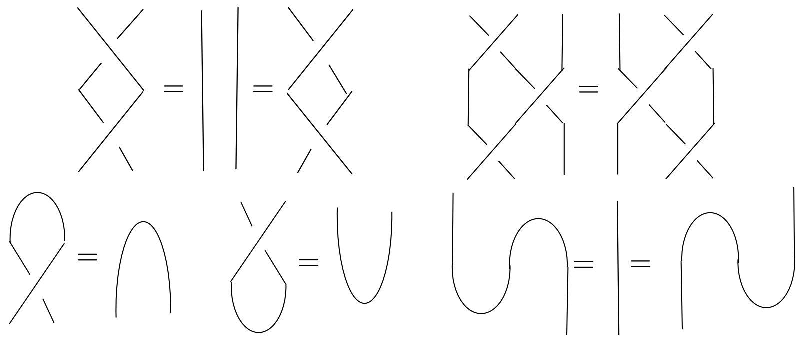
We also denote as p the object of Tangles consisting of a single point in D^2.

(1) Let F : Tangles $\longrightarrow \mathrm{Vect}_{\mathbb{K}}$ be a monoidal functor. Let $V = F(p), R = F(c_+), R' = F(c_-), \alpha = F(u)$, and $\beta = F(n)$.

Show that the following relations are satisfied:

$$RR' = R'R = \mathrm{id}_{V \otimes V}, \quad (1)$$
$$(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R), \quad (2)$$
$$R\beta = \beta, \quad \alpha R = \alpha, \quad (3)$$
$$(\alpha \otimes id_V)(id_V \otimes \beta) = id_V = (id_V \otimes \alpha)(\beta \otimes id_V). \quad (4)$$

Solution: Since F is a monoidal functor, those equations are consequences of the tangle relations below:



(2) Deduce from Equation (4) above that V is finite dimensional.

Solution: Equation (4) is analoguous to the snake relations for U-tubes in TQFT. We prove the finite dimensionality of V in the same way:

Let $(1)\alpha = \sum_{1\leq i\leq k} v_i \otimes w_i$, where $v_i, w_i \in V$. Then for any $x \in V$,

$$x = (x)id_V = (x)(\alpha \otimes id_V)(id_V \otimes \beta) = \sum_{1\leq i\leq k} (v_i \otimes w_i \otimes x)(id_V \otimes \beta) = \sum_{1\leq i\leq k} \beta(w_i, x)v_i,$$

so $v_1, \ldots, v_k$ span V, and V is finite dimensional.

(3) We call a link with $n \geq 1$ components an isotopy class of embeddings of $\coprod_{i=1}^{n} S^1$ in the interior of $D^2 \times [0,1]$.

Explain why a monoidal functor F : Tangles $\longrightarrow$ $\mathrm{Vect}_{\mathbb{K}}$ induces a $\mathbb{K}$-valued invariant of links.

Solution: A link L is a tangle from $(D^2, \emptyset)$ to $(D^2, \emptyset)$, which is the monoidal unit of the category Tangles. Therefore its image by the monoidal functor F is a map $F(L) : \mathbb{K} \longrightarrow \mathbb{K}$, that is $F(L)$ is an element of $\mathbb{K}$. Also by definition of the category of tangles, $F(L)$ depends only on the isotopy class of L, so it is a $\mathbb{K}$-valued invariant of links.